Equitable and Optimal Transport with Multiple Agents

M. Scetbon*   L. Meunier*   J. Atif   M. Cuturi

ENSAE IP PARIS   FACEBOOK AI   Dauphine UNIVERSITÉ PARIS | PSL   Google AI

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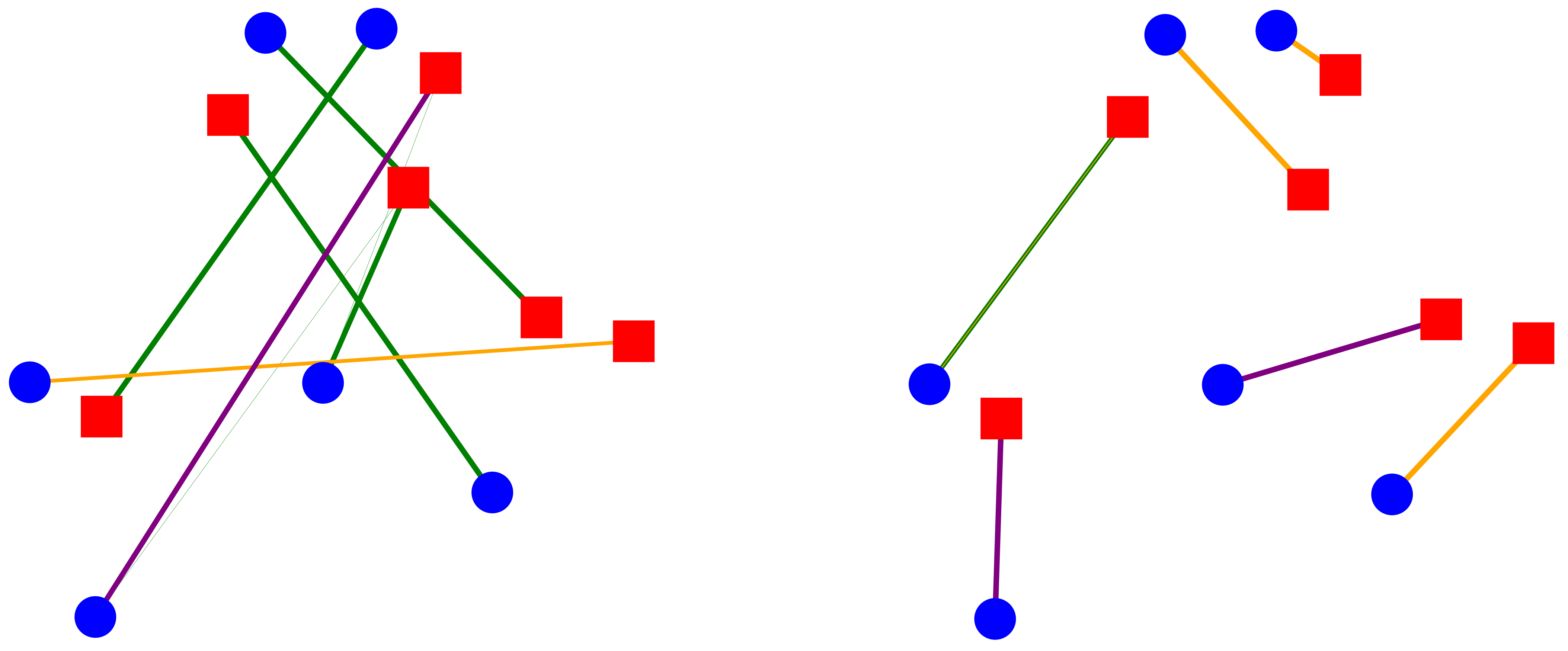
* Authors contributed equally.

Problem: how to split the transportation task in order to obtain an equitable and optimal transportation strategy between multiple agents ?

Primal Formulation



Utility functions: $\forall i,\ c_i < 0$

Cost functions: $\forall i,\ c_i \geq 0$

Distributions: $\mu \in \mathcal{M}_1^+(\mathcal{X})$ and $\nu \in \mathcal{M}_1^+(\mathcal{Y})$

Cost/Utility functions : $c_i : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ for $i \in \{1, \ldots, N\}$

Couplings: $\Gamma^N_{\mu,\nu} := \left\{ (\gamma_i)_{i=1}^N \text{ s.t. } \Pi_{1\sharp} \sum_{i=1}^N \gamma_i = \mu \ , \ \Pi_{2\sharp} \sum_{i=1}^N \gamma_i = \nu \right\}$

Definition of Equitable and Optimal Transport

$$\mathrm{EOT}_{\mathbf{c}}(\mu, \nu) := \inf_{(\gamma_i)_{i=1}^N \in \Gamma^N_{\mu,\nu}} \ \max_{i \in \{1,\ldots,N\}} \int c_i(x, y) d\gamma_i(x, y) \ .$$

- The division of the transportation task is equitable:

Proposition

If the cost/utility functions are of constant sign then we have

$$\mathrm{EOT}_{\mathbf{c}}(\mu,\nu) = \min_{(\gamma_i)_{i=1}^N \in \Gamma^N_{\mu,\nu}} \left\{ t \text{ s.t. } \forall i \in \{1,\ldots,N\} \ \int c_i(x,y)d\gamma_i(x,y) = t \right\}$$

- The division of the transportation task is optimal:

Proposition

If the cost/utility functions are of constant sign then for any $(\gamma_i^*)_{i=1}^N \in \Gamma^N_{\mu,\nu}$ solution of EOT, we have for all $i \in \{1,\ldots,N\}$:

$$\gamma_i^* \in \mathrm{argmin}_{\gamma \in \Gamma^1_{\mu_i^*,\nu_i^*}} \int c_i d\gamma \ \text{ where } \ \mu_i^* := \Pi_{1\sharp}\gamma_i^* \ , \ \nu_i^* := \Pi_{2\sharp}\gamma_i^*$$

Application: EOT solves a relaxation of the fair cake-cutting problem.

Dual Formulation

Dual space: $\mathcal{F}_{\mathbf{c}}^{\lambda} := \{(f, g) \in \mathcal{C}^b(\mathcal{X}) \times \mathcal{C}^b(\mathcal{Y}) \text{ s.t. } \forall i \in \{1,\ldots,N\},\ f \oplus g \leq \lambda_i c_i\}$

Strong duality holds:

$$\mathrm{EOT}_{\mathbf{c}}(\mu, \nu) = \sup_{\substack{\lambda \in \Delta_N^+ \\ (f,g) \in \mathcal{F}_{\mathbf{c}}^{\lambda}}} \int f d\mu + \int g d\nu$$

Link with other Probability Metrics

- If $N = 1$, $\mathrm{EOT}_{\mathbf{c}}(\mu, \nu) = W_c(\mu, \nu)$ where $W_c(\mu, \nu)$ is the OT cost between μ and ν
- If $c_1 = d$ and $c_2 = 2 \times \mathbf{1}_{x \neq y}$, $\mathrm{EOT}_{\mathbf{c}}(\mu, \nu) = \sup_{f \in B_d(\mathcal{X})} \int_{\mathcal{X}} f d\mu - \int_{\mathcal{X}} f d\nu$

  where $B_d(\mathcal{X}) := \left\{ f \in C^b(\mathcal{X}) : \|f\|_\infty + \|f\|_{\mathrm{lip}} \leq 1 \right\}$

Dudley Metric

Entropic Relaxation

Generalized Kullback-Leibler divergence: $\mathrm{KL}(\mu \,||\, \nu) = \int \log \frac{d\mu}{d\nu} d\mu + \int d\nu - \int d\mu$

Definition of the entropic version of EOT:

$$\mathrm{EOT}_{\mathbf{c}}^{\epsilon}(\mu, \nu) := \inf_{(\gamma_i)_{i=1}^{N} \in \Gamma_{\mu,\nu}^{N}} \max_{i \in \{1,\ldots,N\}} \int c_i(x,y) d\gamma_i(x,y) + \epsilon \sum_{j=1}^{N} \mathrm{KL}(\gamma_j \,||\, \mu \otimes \nu)$$

Proposed Algorithm:

Algorithm 1 Projected Alternating Maximization

Input: $\mathbf{C} = (C_i)_{1 \le i \le N}$, a, b, ε, L_λ

Init: $f^0 \leftarrow \mathbf{1}_n$; $g^0 \leftarrow \mathbf{1}_m$; $\lambda^0 \leftarrow (1/N, \ldots, 1/N) \in \mathbb{R}^N$

for $k = 1, 2, \ldots$ **do**

$K^k \leftarrow \sum_{i=1}^{N} K_i^{\lambda_i^{k-1}}$,

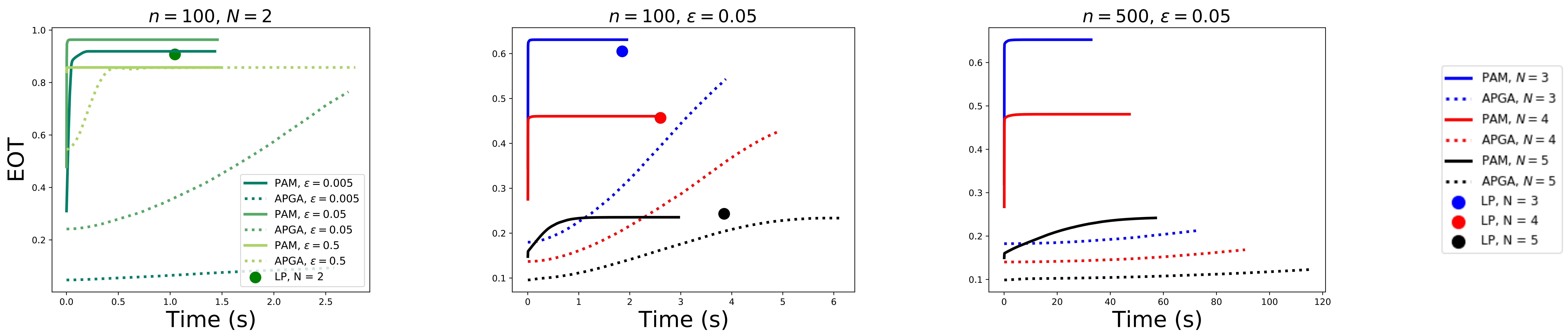
$c_k \leftarrow \langle f^{k-1}, K^k g^{k-1} \rangle$, $f^k \leftarrow \frac{c_k a}{K^k g^{k-1}}$,

$d_k \leftarrow \langle f^k, K^k g^{k-1} \rangle$, $g^k \leftarrow \frac{d_k b}{(K^k)^T f^k}$,

$\lambda^k \leftarrow \mathrm{Proj}_{\Delta_N^+}\left(\lambda^{k-1} + \frac{1}{L_\lambda} \nabla_\lambda F_{\mathbf{C}}^{\varepsilon}(\lambda^{k-1}, f^k, g^k)\right)$.

end

Result: λ, f, g



Other Applications of EOT

- Minimal transportation time
- Sequential OT

Thank you