ENSAE IP PARIS
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Equitable and Optimal Transport with Multiple Agents

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Overview

Problem: how to split the transportation task in order to obtain an equitable and optimal transportation strategy between multiple agents ?

Contributions:

- We introduce *EOT* (**E**quitable and **O**ptimal **T**ransport) and show that it solves a *fair division problem* where heterogeneous resources have to be shared among *multiple agents*.
- We derive its dual and prove *strong duality* results. As a by-product, we show that EOT is related to some usual IPMs families and in particular the widely known *Dudley metric*.
- We propose an *entropic regularized version* of the problem, derive its dual formulation, obtain strong duality. We then derive *an efficient algorithm* to compute it.
- We propose other applications of EOT for *Operations Research problems.*

Primal Formulation

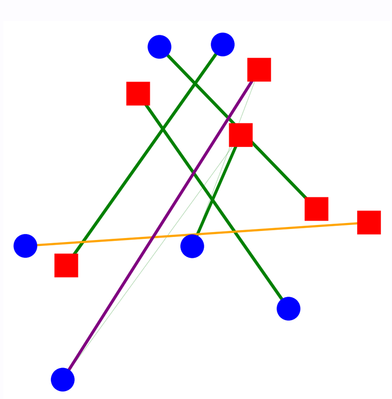
Distributions: $\mu \in \mathcal{M}_1^+(\mathcal{X})$ and $\nu \in \mathcal{M}_1^+(\mathcal{Y})$

Cost/Utility functions : $c_i : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ for $i \in \{1,\dots,N\}$

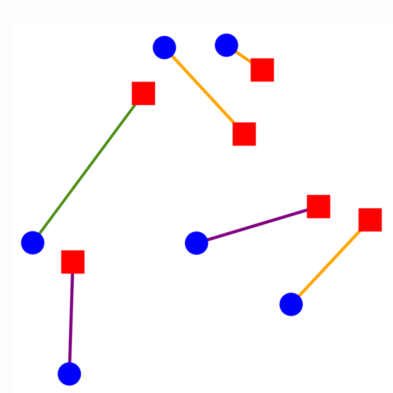
Couplings: $\Gamma^N_{\mu,\nu} := \left\{ (\gamma_i)_{i=1}^N \text{ s.t. } \Pi_{1\sharp}\sum_{i=1}^N \gamma_i = \mu \text{ , } \Pi_{2\sharp}\sum_{i=1}^N \gamma_i = \nu \right\}$

Definition of Equitable and Optimal Transport

$$\text{EOT}_{\mathbf{c}}(\mu,\nu) := \inf_{(\gamma_i)_{i=1}^N \in \Gamma^N_{\mu,\nu}} \max_{i\in\{1,\dots,N\}} \int c_i(x,y)d\gamma_i(x,y) \text{ .}$$



Utility functions:
$\forall i,\ c_i < 0$



Cost functions:
$\forall i,\ c_i \geq 0$

Equitable and Optimal

- The division of the transportation task is equitable:

Proposition
If the cost/utility functions are of constant sign then we have

$$\text{EOT}_{\mathbf{c}}(\mu,\nu) = \min_{(\gamma_i)_{i=1}^N \in \Gamma^N_{\mu,\nu}} \left\{ t \text{ s.t. } \forall i \in \{1,\dots,N\} \quad \int c_i(x,y)d\gamma_i(x,y) = t \right\}$$

- The division of the transportation task is optimal:

Proposition
If the cost/utility functions are of constant sign then for any $(\gamma_i^*)_{i=1}^N \in \Gamma^N_{\mu,\nu}$ solution of EOT, we have for all $i \in \{1,\dots,N\}$:

$$\gamma_i^* \in \text{argmin}_{\gamma \in \Gamma^1_{\mu_i^*,\nu_i^*}} \int c_i d\gamma \text{ where } \mu_i^* := \Pi_{1\sharp}\gamma_i^* \text{ , } \nu_i^* := \Pi_{2\sharp}\gamma_i^*$$

Applications:

EOT solves a relaxation of the fair cake-cutting problem.

Dual Formulation

Dual space:

$$\mathcal{F}_{\mathbf{c}}^\lambda := \{(f,g) \in \mathcal{C}^b(\mathcal{X}) \times \mathcal{C}^b(\mathcal{Y}) \text{ s.t. } \forall i \in \{1,\dots,N\}, f \oplus g \leq \lambda_i c_i\}$$

Strong duality holds: $$\text{EOT}_{\mathbf{c}}(\mu,\nu) = \sup_{\substack{\lambda \in \Delta_N^+ \\ (f,g) \in \mathcal{F}_{\mathbf{c}}^\lambda}} \int f d\mu + \int g d\nu$$

Link with other Probability Metrics:

- Optimal Transport: if $N=1$, $\text{EOT}_{\mathbf{c}}(\mu,\nu) = W_c(\mu,\nu)$
- Dudley Metric: if $c_1 = d$ and $c_2 = 2 \times \mathbf{1}_{x \neq y}$,

$$\text{EOT}_{\mathbf{c}}(\mu,\nu) = \sup_{f \in B_d(\mathcal{X})} \int_{\mathcal{X}} f d\mu - \int_{\mathcal{X}} f d\nu$$

$$B_d(\mathcal{X}) := \left\{ f \in C^b(\mathcal{X}) \colon \|f\|_\infty + \|f\|_{\text{lip}} \leq 1 \right\}$$

Entropic Relaxation

KL divergence: $\text{KL}(\mu\,||\,\nu) = \int \log \frac{d\mu}{d\nu} d\mu + \int d\nu - \int d\mu$

Definition of the entropic version of EOT:

$$\text{EOT}^\epsilon_{\mathbf{c}}(\mu,\nu) := \inf_{(\gamma_i)_{i=1}^N \in \Gamma^N_{\mu,\nu}} \max_{i\in\{1,\dots,N\}} \int c_i(x,y)d\gamma_i(x,y) + \epsilon \sum_{j=1}^N \text{KL}(\gamma_j\,||\,\mu \otimes \nu)$$

Strong duality holds:

$$\text{EOT}^\epsilon_{\mathbf{c}}(\mu,\nu) = \sup_{\lambda \in \Delta_N^+} \sup_{\substack{f \in \mathcal{C}_b(\mathcal{X}) \\ g \in \mathcal{C}_b(\mathcal{Y})}} \int f d\mu + \int g d\nu - \epsilon \sum_{i=1}^N \left(\int e^{\frac{f(x)+g(y)-\lambda_i c_i(x,y)}{\epsilon}} d\mu(x)d\nu(y) - 1 \right)$$

Projected Sinkhorn Algorithm

Algorithm 1 Projected Alternating Maximization

Input: $\mathbf{C} = (C_i)_{1\leq i\leq N}$, a, b, ε, L_λ
Init: $f^0 \leftarrow \mathbf{1}_n$; $g^0 \leftarrow \mathbf{1}_m$; $\lambda^0 \leftarrow (1/N,\dots,1/N) \in \mathbb{R}^N$
for $k = 1,2,\dots$ **do**
  $K^k \leftarrow \sum_{i=1}^N K_i^{\lambda_i^{k-1}}$,
  $c_k \leftarrow \langle f^{k-1}, K^k g^{k-1}\rangle$, $f^k \leftarrow \frac{c_k a}{K^k g^{k-1}}$,
  $d_k \leftarrow \langle f^k, K^k g^{k-1}\rangle$, $g^k \leftarrow \frac{d_k b}{(K^k)^T f^k}$,
  $\lambda^k \leftarrow \text{Proj}_{\Delta_N^+}\left(\lambda^{k-1} + \frac{1}{L_\lambda}\nabla_\lambda F^\varepsilon_{\mathbf{C}}(\lambda^{k-1}, f^k, g^k)\right)$.
end
Result: λ, f, g

Operations Research

- Minimal transportation time
- Sequential optimal transport